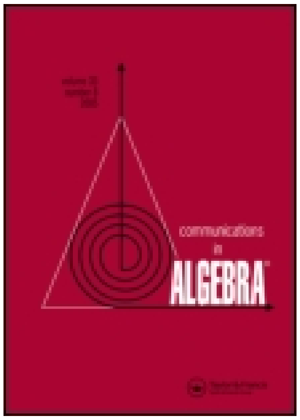

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GORENSTEIN PROJECTIVE DIMENSION RELATIVE TO A SEMIDUALIZING BIMODULE

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Let S and R be rings and $_SC_R$ a semidualizing bimodule. We investigate the relation between the G_C-syzygy with the C-syzygy of a module as well as the relation between the G_C-projective resolution and the projective resolution of a module. As a consequence, we get that if

$$\mathbb{G} : \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

is an exact sequence of S-modules with all G_i, G^i G_C-projective, such that $\mathrm{Hom}_S(\mathbb{G}, T)$ is still exact for any module T which is isomorphic to a direct summand of direct sums of copies of $_SC$, then $\mathrm{Im}(G_0 \to G^0)$ is also G_C-projective. We obtain a criterion for computing the G_C-projective dimension of modules. When $_SC_R$ is a faithfully semidualizing bimodule, we study the Foxby equivalence between the subclasses of the Auslander class and that of the Bass class with respect to C.

Key Words: Auslander class; Bass class; Foxby equivalence; (faithfully) semidualizing bimodules; G_C-projective dimension; G_C-projective modules.

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1. INTRODUCTION

Auslander and Bridger introduced in [2] the G-dimension for finitely generated modules over Noetherian rings. Then Enochs and Jenda introduced in [7] the Gorenstein projective dimension for arbitrary modules over a general ring, which is a generalization of the G-dimension. The homological properties of the Gorenstein projective dimension and some generalized versions of such a dimension have been studied by many authors, see [1, 3, 8–12, 14–17] and the literatures listed in them. White introduced in [17] the G_C-projective modules and gave a functorial description of the G_C-projective dimension of modules with respect to a semidualizing module C over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [17]. Over a commutative Noetherian ring, the G_C-projective modules and the G_C-projective

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dimension were called C-Gorenstein projective modules and the C-Gorenstein projective dimension in [11], respectively. Note that the noncommutative versions of almost all the results in [17] also hold true. Based on these results, in this article we further investigate the properties of the G_C-projective modules and the G_C-projective dimension of modules over general rings.

This article is organized as follows.

In Section 2, we establish the relation between the G_C-syzygy with the C-syzygy of a module as well as the relation between the G_C-projective resolution and the projective resolution of a module. As a consequence, we get that an iteration of the procedure used to define the G_C-projective modules yields exactly the G_C-projective modules. Then we obtain a new equivalent characterization of G_C-projective modules, by which it yields that the G_C-projective modules possess the symmetry just as the Gorenstein projective modules do.

In Section 3, we get some properties of G_C-projective dimension of modules. In particular, as an application of the results obtained in Section 2, we get a criterion for compute such a dimension. Let S, R be rings and ${}_SC_R$ a semidualizing bimodule, and let M be a left S-module and $n \geq 0$. We prove that the G_C-projective dimension of M is at most n if and only if for every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\operatorname{Mod} S$ such that X_t is G_C-projective and X_i is projective or isomorphic to a direct summand of a direct sum of copies of ${}_SC$ for $i \neq t$. Let ${}_SC_R$ be a faithfully semidualizing bimodule and $n \geq 0$. We prove that if S (resp., R) is a n-Gorenstein ring, then $\operatorname{id}_S C \leq n$ (resp., $\operatorname{id}_{R^{op}} C \leq n$) if and only if every left S-module (resp., every right R-module) has G_C-projective dimension at most n.

In Section 4, we prove that if ${}_SC_R$ a faithfully semidualizing bimodule, then for any $n \geq 0$, the class of left R-modules with Gorenstein projective dimension (resp., G_C-injective dimension) at most n in the Auslander class $\mathscr{A}_C(R)$ and the class of left S-modules with G_C-projective dimension (resp., Gorenstein injective dimension) at most n in the Bass class $\mathscr{B}_C(S)$ are equivalent under Foxby equivalence.

2. G_C-PROJECTIVE MODULES

Throughout this article, S and R are associative with identity, and all modules are unitary. We use $\operatorname{Mod} S$ (resp., $\operatorname{Mod} R^{op}$) to denote the class of left S-modules (resp., right R-modules).

At the beginning of this section, we recall some notions from [12].

Definition 2.1 ([12]). An (S, R)-bimodule $C = {}_SC_R$ is called *semidualizing* if the following conditions are satisfied:

(a1) ${}_SC$ admits a degreewise finite S-projective resolution;
(a2) C_R admits a degreewise finite R^{op}-projective resolution;
(b1) The homothety map ${}_SS_S \xrightarrow{Sr} \operatorname{Hom}_{R^{op}}(C, C)$ is an isomorphism;
(b2) The homothety map ${}_RR_R \xrightarrow{rR} \operatorname{Hom}_S(C, C)$ is an isomorphism;
(c1) $\operatorname{Ext}^i_S(C, C) = 0$ for any $i \geq 1$;
(c2) $\operatorname{Ext}^i_{R^{op}}(C, C) = 0$ for any $i \geq 1$.

From now on, ${}_SC_R$ is a semidualizing bimodule.

Definition 2.2 ([12]). A module in $\mathrm{Mod}\, S$ is called *C-projective* if it has the form $C \otimes_R P$ for some projective module $P \in \mathrm{Mod}\, R$. A module in $\mathrm{Mod}\, R$ is called *C-injective* if it has the form $\mathrm{Hom}_S(C, I)$ for some injective module $I \in \mathrm{Mod}\, S$. Set

$$\mathscr{P}_C(S) = \{C \otimes_R P \mid {}_RP \text{ is projective}\}, \qquad \text{and}$$
$$\mathscr{I}_C(R) = \{\mathrm{Hom}_S(C, I) \mid {}_SI \text{ is injective}\}.$$

Definition 2.3 ([12]). The *Auslander class* $\mathscr{A}_C(R)$ with respect to C consists of all module $M \in \mathrm{Mod}\, R$ satisfying:

(A1) $\mathrm{Tor}_i^R(C, M) = 0$ for any $i \geq 1$;
(A2) $\mathrm{Ext}_S^i(C, C \otimes_R M) = 0$ for any $i \geq 1$; and
(A3) The natural evaluation homomorphism $\mu_M : M \to \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism (of R-modules).

The *Bass class* $\mathscr{B}_C(S)$ with respect to C consists of all modules $N \in \mathrm{Mod}\, S$ satisfying:

(B1) $\mathrm{Ext}_S^i(C, N) = 0$ for any $i \geq 1$;
(B2) $\mathrm{Tor}_i^R(C, \mathrm{Hom}_S(C, N)) = 0$ for any $i \geq 1$; and
(B3) The natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_S(C, N) \to N$ is an isomorphism (of S-modules).

Let $M \in \mathrm{Mod}\, S$. We denote $\mathrm{Add}_S M$ (resp., $\mathrm{Prod}_S M$) the subclass of $\mathrm{Mod}\, S$ consisting of all modules isomorphic to direct summands of direct sums (resp., direct products) of copies of M. The following result was proved in [9, Theorem 3.1]. We give it a quick proof.

Proposition 2.4.

(1) $\mathscr{P}_C(S) = \mathrm{Add}_S C$.
(2) $\mathscr{I}_C(R) = \mathrm{Prod}_R C^+$, *where* $C^+ = \mathrm{Hom}_S(C, E)$ *with* ${}_SE$ *an injective cogenerator for* $\mathrm{Mod}\, S$.

Proof. (1) It is clear that $\mathscr{P}_C(S) \subseteq \mathrm{Add}_S C$. Now we show $\mathrm{Add}_S C \subseteq \mathscr{P}_C(S)$. For any $M \in \mathrm{Add}_S C$, there exists $N \in \mathrm{Mod}\, S$ such that $M \oplus N \cong C^{(J)}$ for some cardinal J. Note that $\mathscr{B}_C(S)$ is closed under direct sums and direct summands by [12, Proposition 4.2]. Since $C \cong C \otimes_R R \in \mathscr{B}_C(S)$ by [12, Lemma 5.1], both $C^{(J)}$ and M are in $\mathscr{B}_C(S)$. Since $\mathrm{Hom}_S(C, M) \oplus \mathrm{Hom}_S(C, N) \cong \mathrm{Hom}_S(C, C^{(J)}) \cong R^{(J)}$, $\mathrm{Hom}_S(C, M) \in \mathrm{Mod}\, R$ is projective. Thus $M \in \mathscr{P}_C(S)$ by [12, Lemma 5.1].

(2) Let $I \in \mathrm{Mod}\, S$ be injective. Then I is isomorphic to a direct summand of E^J for some cardinal J. So $\mathrm{Hom}_S(C, I)$ is isomorphic to a direct summand of $\mathrm{Hom}_S(C, E^J)(\cong (C^+)^J)$, and $\mathrm{Hom}_S(C, I) \in \mathrm{Prod}_R C^+$. Thus we have $\mathscr{I}_C(R) \subseteq \mathrm{Prod}_R C^+$.

In what follows, we show $\mathrm{Prod}_R C^+ \subseteq \mathscr{I}_C(R)$. For any $X \in \mathrm{Prod}_R C^+$, there exists $Y \in \mathrm{Mod}\, R$ such that $X \oplus Y \cong (C^+)^J$ for some cardinal J. Note that $\mathscr{A}_C(R)$ is closed under direct products and direct summands by [12, Proposition 4.2]. Since

$C^+ \in \mathscr{A}_C(R)$ by [12, Lemma 5.1], both $(C^+)^J$ and X are in $\mathscr{A}_C(R)$. Since ${}_SE \in \mathscr{B}_C(S)$ by [12, Lemma 4.1], $(C \otimes_R X) \oplus (C \otimes_R Y) \cong C \otimes_R (C^+)^J \cong E^J \in \mathscr{B}_C(S)$. Thus $C \otimes_R X \in \operatorname{Mod} S$ is injective. So $X \in \mathscr{I}_C(R)$ by [12, Lemma 5.1]. $\square$

Let $\mathscr{C}$ be a subclass of $\operatorname{Mod} S$. Recall that a sequence $\mathbf{L}$ in $\operatorname{Mod} S$ is called $\operatorname{Hom}_S(-, \mathscr{C})$ (resp., $\operatorname{Hom}_S(\mathscr{C}, -)$) exact if the sequence $\operatorname{Hom}_S(\mathbf{L}, C')$ (resp., $\operatorname{Hom}_S(C', \mathbf{L})$) is exact for any $C' \in \mathscr{C}$. Note that $\mathscr{P}_C(S) = \operatorname{Add}_S C$ and $\mathscr{I}_C(R) = \operatorname{Prod} C^+$ by Proposition 2.4. The following notions were introduced by Holm and Jørgensen in [11] and White in [17] for commutative rings. We give the noncommutative versions of them.

Definition 2.5.

(1) A *complete $\mathscr{P}\mathscr{P}_C$-resolution* is a $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact exact sequence:

$$\mathbf{X} = \cdots \to P_1 \to P_0 \to C^0 \to C^1 \to \cdots \tag{2.1}$$

in $\operatorname{Mod} S$ with all P_i projective and $C^i \in \operatorname{Add}_S C$. A module $M \in \operatorname{Mod} S$ is called *G_C-projective* if there exists a complete $\mathscr{P}\mathscr{P}_C$-resolution as in (2.1) with $M \cong \operatorname{Im}(P_0 \to C^0)$. Set

$$\mathscr{G}\mathscr{P}_C(S) = \text{the class of } G_C\text{-projective modules in } \operatorname{Mod} S.$$

(2) A *complete $\mathscr{I}_C\mathscr{I}$-resolution* is a $\operatorname{Hom}_R(\operatorname{Prod} C^+, -)$ exact exact sequence:

$$\mathbf{Y} = \cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots \tag{2.2}$$

in $\operatorname{Mod} R$ with all I^i injective and $I_i \in \operatorname{Prod}_R C^+$. A module $M \in \operatorname{Mod} R$ is called *G_C-injective* if there exists a complete $\mathscr{I}_C\mathscr{I}$-resolution as in (2.2) with $M \cong \operatorname{Im}(I_0 \to I^0)$. Set

$$\mathscr{G}\mathscr{I}_C(R) = \text{the class of } G_C\text{-injective modules in } \operatorname{Mod} R.$$

It is trivial that in case ${}_SC_R = {}_SS_S$, the G_C-projective modules and G_C-injective modules are just the usual Gorenstein projective modules and Gorenstein injective modules, respectively.

Note

(1) *In* [17] *the rings are commutative rings, but the noncommutative analogs of all the results from* 1.1 *to* 4.4 *in it are also valid. So in the following, we will cite these results directly in our setting.*
(2) *In what follows, we only deal with the G_C-projectivity of modules. But it should be pointed out that all of the obtained results have a G_C-injective counterpart by using completely dual arguments.*

Lemma 2.6. *Let $M \in \operatorname{Mod} S$ be G_C-projective. Then there exist* $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ *exact sequences*

$$0 \to M \to G \to N \to 0$$

and

$$0 \to K \to P \to M \to 0$$

in Mod S *with* N, K G_C*-projective,* $G \in \mathrm{Add}_S C$*, and* P *projective.*

Proof. By the definition of G_C-projective modules and [17, Proposition 2.9]. □

For any $M \in \mathrm{Mod}\, S$ and $n \geq 1$, we denote $\mathrm{Ext}_S^n(M, \mathrm{Add}_S C) = \{\mathrm{Ext}_S^n(M, C') \mid C' \in \mathrm{Add}_S C\}$. The following result plays a crucial role in the rest of this section.

Lemma 2.7. *Let*

$$0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0 \tag{2.3}$$

be an exact sequence in Mod S *with* G_0, G_1 G_C*-projective. Then*:

(1) *We have the following exact sequences*

$$0 \to A \to C_1 \to G \to M \to 0 \tag{2.4}$$

and

$$0 \to A \to H \to P \to M \to 0 \tag{2.5}$$

with $C_1 \in \mathrm{Add}_S C$*,* P *projective, and* G, H G_C*-projective.*

(2) *If the exact sequence* (2.3) *is* $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ *exact, then so are* (2.4) *and* (2.5).

Proof. (1) Since G_1 is G_C-projective, there exists an exact sequence $0 \to G_1 \to C_1 \to G' \to 0$ with $C_1 \in \mathrm{Add}_S C$ and G' G_C-projective by Lemma 2.6. Then we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_1 & \longrightarrow & B & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & B & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0.\\
 & & \downarrow & & \downarrow & & & & \\
 & & G' & = & G' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since G_0 and G' are G_C-projective, G is also G_C-projective by [17, Theorem 2.8]. Connecting the middle rows in the above two diagrams, we get the first desired exact sequence (2.4).

Since G_0 is G_C-projective, there exists an exact sequence $0 \to G'' \to P \to G_0 \to 0$ with P projective and G'' G_C-projective by Lemma 2.6. Then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & G'' & = & G'' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0.\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Consider the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G'' & = & G'' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_1 & \longrightarrow & \mathrm{Im}\, f & \longrightarrow & 0.\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since G_1 and G'' are G_C-projective, H is also G_C-projective by [17, Theorem 2.8]. Connecting the middle rows in the above two diagrams, we get the second desired exact sequence (2.5).

(2) Note that $\mathrm{Ext}^i_S(L, \mathrm{Add}_S C) = 0$ for any G_C-projective module $L \in \mathrm{Mod}\, S$ and $i \geq 1$ by [17, Proposition 2.2]. If the exact sequence (2.3) is $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact, then $\mathrm{Ext}^1_S(M, \mathrm{Add}_S C) = 0 = \mathrm{Ext}^2_S(M, \mathrm{Add}_S C)$ and $\mathrm{Ext}^1_S(\mathrm{Im}\, f, \mathrm{Add}_S C) = 0$. So in the proof of (1), $\mathrm{Ext}^1_S(B, \mathrm{Add}_S C) = 0$ and $\mathrm{Ext}^1_S(N, \mathrm{Add}_S C) = 0$. Thus the exact sequences (2.4) and (2.5) are $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact. □

The following result establishes the relation between the G_C-syzygy with the C-syzygy of a module as well as the relation between the G_C-projective resolution and the projective resolution of a module.

Lemma 2.8 (cf. [17, Theorem 3.6] and Proposition 3.5 below). *Let $n \geq 1$ and*

$$0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0 \tag{2.6}$$

be an exact sequence in $\mathrm{Mod}\, S$ *with all G_i G_C-projective. Then we have the following*:

(1) *There exist exact sequences*:

$$0 \to A \to C_{n-1} \to \cdots \to C_1 \to C_0 \to N \to 0 \tag{2.7}$$

and

$$0 \to M \to N \to G \to 0$$

in $\mathrm{Mod}\, S$ *with all $C_i \in \mathrm{Add}_S C$ and G G_C-projective.*

(2) *There exist exact sequences*

$$0 \to B \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0 \tag{2.8}$$

and

$$0 \to H \to B \to A \to 0$$

in $\mathrm{Mod}\, S$ *with all P_i projective and H G_C-projective.*

(3) *If the exact sequence* (2.6) *is* $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ *exact, then so are* (2.7) *and* (2.8).

Proof. We proceed by induction on n.

(1) When $n = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in $\mathrm{Mod}\, S$. Since we have a $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequence $0 \to G_0 \to C_0 \to G \to 0$ in $\mathrm{Mod}\, S$ with $C_0 \in \mathrm{Add}_S C$ and G G_C-projective by Lemma 2.6, we have the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in $\mathrm{Mod}\, S$ with all G_i G_C-projective. Put $K = \mathrm{Coker}(G_{n-1} \to G_{n-2})$. By Lemma 2.7, we get an exact sequence

$$0 \to A \to C_{n-1} \to G'_{n-2} \to K \to 0 \tag{2.9}$$

in $\mathrm{Mod}\, S$ with $C_{n-1} \in \mathrm{Add}_S C$ and G'_{n-2} G_C-projective. Put $A' = \mathrm{Im}(C_{n-1} \to G'_{n-2})$. Then we get an exact sequence $0 \to A' \to G'_{n-2} \to G_{n-3} \to \cdots \to G_1 \to G_0 \to M \to 0$ in $\mathrm{Mod}\, S$. So, by the induction hypothesis, we get the assertion.

(2) When $n = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in $\mathrm{Mod}\, S$. Since we have a $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequence $0 \to H \to P_0 \to G_0 \to 0$ in $\mathrm{Mod}\, S$ with P_0 projective and H G_C-projective by Lemma 2.6, then we have the following pullback diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & B & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

The middle row and the first column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in $\operatorname{Mod} S$ with all G_i G_C-projective. Put $K = \operatorname{Ker}(G_1 \to G_0)$. By Lemma 2.7, we get an exact sequence

$$0 \to K \to G_1' \to P_0 \to M \to 0 \tag{2.10}$$

in $\operatorname{Mod} S$ with P_0 projective and G_1' G_C-projective. Put $M' = \operatorname{Im}(G_1' \to P_0)$. Then we get an exact sequence $0 \to A \to G_{n-1} \to \cdots \to G_2 \to G_1' \to M \to 0$ in $\operatorname{Mod} S$. So, by the induction hypothesis, we get the assertion.

(3) If the exact sequence (2.6) is $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact, then the middle rows in the above two commutative diagrams are also $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact. On the other hand, we can choose both (2.9) and (2.10) to be $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact by Lemma 2.7. Then we get the assertion by the induction hypothesis. □

We denote $\mathcal{G}^2\mathcal{P}_C(S) = \{A \in \operatorname{Mod} S \mid$ there exists a $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact exact sequence $\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ in $\operatorname{Mod} S$ with all G_i and G^i in $\mathcal{GP}_C(S)$ and $A \cong \operatorname{Im}(G_0 \to G^0)\}$. We denote $\overline{\operatorname{Add}_S C} = \operatorname{Add}_S C \bigcup \operatorname{Add}_S S$.

The following result means that an iteration of the procedure used to define the G_C-projective modules yields exactly the G_C-projective modules, which generalizes [14, Theorem A].

Theorem 2.9. $\mathcal{G}^2\mathcal{P}_C(S) = \mathcal{GP}_C(S)$.

Proof. By [17, Proposition 2.6], we have $\overline{\operatorname{Add}_S C} \subseteq \mathcal{GP}_C(S)$. So $\mathcal{GP}_C(S) \subseteq \mathcal{G}^2\mathcal{P}_C(S)$. In the following, we prove the converse inclusion.

Let

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

be a $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact exact sequence in $\operatorname{Mod} S$ with all G_i and G^i in $\mathcal{GP}_C(S)$ and $A \cong \operatorname{Im}(G_0 \to G^0)$. Then $\operatorname{Ext}^i_S(A, \operatorname{Add}_S C) = 0$ for any $i \geq 1$.

Put $A^i = \operatorname{Im}(G^i \to G^{i+1})$ for any $i \geq 0$. By Lemma 2.8, there exist exact sequences

$$0 \to A \to C^0 \to N^0 \to 0$$

and

$$0 \to A^0 \to N^0 \to G \to 0$$

in $\operatorname{Mod} S$ with $C^0 \in \operatorname{Add}_S C$ and G G_C-projective such that the former one is $\operatorname{Hom}_S(-, \operatorname{Add}_S C)$ exact. Consider the following pushout diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & A^0 & \longrightarrow & N^0 & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G^1 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & & & \\ & & A^1 & = & A^1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

By [17, Theorem 2.8] and the exactness of the middle row in the above diagram, G' is G_C-projective. Because the first column in the above diagram is $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact, $\mathrm{Ext}^1_S(A^1, \mathrm{Add}_S C) = 0$. It yields that the middle column in the above diagram is also $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact, and so we get a $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequence

$$0 \to N^0 \to G' \to G^2 \to G^3 \to \cdots$$

in $\mathrm{Mod}\, S$. Then by the above argument, we have $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequences

$$0 \to N^0 \to C^1 \to N^1 \to 0$$

and

$$0 \to N^1 \to G'' \to G^3 \to G^4 \to \cdots$$

in $\mathrm{Mod}\, S$. We proceed in this manner to get a $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequence

$$0 \to A \to C^0 \to C^1 \to \cdots$$

in $\mathrm{Mod}\, S$ with all $C^i \in \mathrm{Add}_S C$. Thus A is G_C-projective by [17, Proposition 2.2], and therefore, $\mathcal{G}^2\mathcal{P}_C(S) \subseteq \mathcal{G}\mathcal{P}_C(S)$. The proof is finished. □

By Theorem 2.9, we get immediately the following equivalent characterization of G_C-projective modules. It shows that the G_C-projective modules possess the symmetry just as the Gorenstein projective modules do.

Corollary 2.10. *A module $M \in \mathrm{Mod}\, S$ is G_C-projective if and only if there exists a $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequence*

$$\mathbf{C} : \cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$$

in $\mathrm{Mod}\, S$ with all $C_i, C^i \in \overline{\mathrm{Add}_S C}$ and $M \cong \mathrm{Im}(C_0 \to C^0)$.

3. G_C-PROJECTIVE DIMENSION OF MODULES

The notion of G_C-projective dimension of modules was introduced by White in [17] for commutative rings. Here we give its noncommutative version.

Definition 3.1 ([17]). For a module $M \in \operatorname{Mod} S$, the *$G_C$-projective dimension* of M, denoted by $G_C\text{-pd}_S M$, is defined as $\inf\{n \mid$ there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in $\operatorname{Mod} S$ with all G_i G_C-projective$\}$. We have $G_C\text{-pd}_S M \geq 0$, and we set $G_C\text{-pd}_S M$ infinity if no such integer exists. Dually, the *G_C-injective dimension* of M is defined.

In this section, we investigate the properties of G_C-projective dimension of modules. We begin with the following standard result.

Lemma 3.2. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* $\operatorname{Mod} S$.

(1) $G_C\text{-pd}_S N \leq \max\{G_C\text{-pd}_S M,\ G_C\text{-pd}_S L + 1\}$, *and the equality holds true if* $G_C\text{-pd}_S M \neq G_C\text{-pd}_S L$.
(2) $G_C\text{-pd}_S L \leq \max\{G_C\text{-pd}_S M,\ G_C\text{-pd}_S N - 1\}$, *and the equality holds true if* $G_C\text{-pd}_S M \neq G_C\text{-pd}_S N$.
(3) $G_C\text{-pd}_S M \leq \max\{G_C\text{-pd}_S L,\ G_C\text{-pd}_S N\}$, *and the equality holds true if* $G_C\text{-pd}_S N \neq G_C\text{-pd}_S L + 1$.

Proof. It is easy to get the assertions by [17, Propositions 2.12 and 2.14]. □

The following result is a G_C-projective version of the corresponding result about projective dimension of modules, which extends [17, Proposition 2.10].

Proposition 3.3. *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence in* $\operatorname{Mod} S$.

(1) *If* $L \neq 0$ *and* N *is* G_C*-projective, then* $G_C\text{-pd}_S L = G_C\text{-pd}_S M$.
(2) *Assume that* $G_C\text{-pd}_S M = n (\geq 0)$. *If* $G_C\text{-pd}_S L \geq n$ *or* $G_C\text{-pd}_S N \geq n + 1$, *then* $G_C\text{-pd}_S L = G_C\text{-pd}_S N - 1$.

Proof. (1) By Lemma 3.2(3).

(2) By [17, Proposition 2.14], we may assume that all of $G_C\text{-pd}_S L$, $G_C\text{-pd}_S M$, and $G_C\text{-pd}_S N$ are finite. Since $G_C\text{-pd}_S M = n$ by assumption, $\operatorname{Ext}^i_S(M, \operatorname{Add}_S C) = 0$ for any $i \geq n+1$ by [17, Proposition 2.12]. Then $\operatorname{Ext}^i_S(L, C') \cong \operatorname{Ext}^{i+1}_S(N, C')$ for any $C' \in \operatorname{Add}_S C$ and $i \geq n+1$. So for any $m \geq n$, we have $G_C\text{-pd}_S L \leq m$ if and only if $G_C\text{-pd}_S N \leq m+1$ again by [17, Proposition 2.12]. Now the assertion follows easily. □

Recall that for a module $M \in \operatorname{Mod} S$, the *$C$-projective dimension of M*, denoted by $C\text{-dim}_S M$, is defined as $\inf\{n \mid$ there exists an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to M \to 0$ in $\operatorname{Mod} S$ with all $C_i \in \operatorname{Add}_S C\}$. We set $C\text{-dim}_S M$ infinity if no such integer exists. The following corollary generalizes [4, Lemma 2.17].

Corollary 3.4. *Let* $M \in \operatorname{Mod} S$ *with* $G_C\text{-pd}_S M = n$. *Then there exists an exact sequence* $0 \to M \to N \to G \to 0$ *in* $\operatorname{Mod} S$ *with* $C\text{-dim}_S N = n$ *and* G G_C*-projective.*

Proof. Let $M \in \mathrm{Mod}\, S$ with $G_C\text{-pd}_S M = n$. Then we use Lemma 2.8(1) with $A = 0$ to get an exact sequence $0 \to M \to N \to G \to 0$ in $\mathrm{Mod}\, S$ with $C\text{-dim}_S\, N \leq n$ and G G_C-projective. By Proposition 3.3(1), $G_C\text{-pd}_S N = n$, and thus $C\text{-dim}_S\, N = n$. □

We give a criterion for computing the G_C-projective dimension of modules as follows. It generalizes [10, Theorem 2.20] and [5, Theorem 3.1].

Proposition 3.5 (cf. [17, Theorem 3.6]). *The following statements are equivalent for any $M \in \mathrm{Mod}\, S$ and $n \geq 0$.*

(1) $G_C\text{-pd}_S M \leq n$.
(2) *For every non-negative integer t such that $0 \leq t \leq n$, there exists an exact sequence $0 \to X_n \to \cdots \to X_1 \to X_0 \to M \to 0$ in $\mathrm{Mod}\, S$ such that X_t is G_C-projective and $X_i \in \overline{\mathrm{Add}_S C}$ for $i \neq t$.*

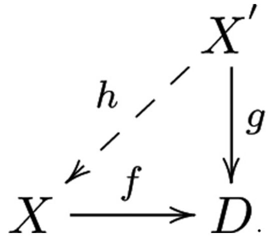
Proof. (2) $\Rightarrow$ (1) It is trivial.

(1) $\Rightarrow$ (2) We proceed by induction on n. Suppose $G_C\text{-pd}_S M \leq 1$. Then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in $\mathrm{Mod}\, S$ with G_0 and G_1 G_C-projective. By Lemma 2.7 with $A = 0$, we get the exact sequences $0 \to C_1 \to G_0' \to M \to 0$ and $0 \to G_1' \to P_0 \to M \to 0$ in $\mathrm{Mod}\, S$ with $C_1 \in \mathrm{Add}_S C$, P_0 projective, and G_0', G_1' G_C-projective.

Now suppose $G_C\text{-pd}_S M = n \geq 2$. Then there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ in $\mathrm{Mod}\, S$ with all G_i G_C-projective for any $0 \leq i \leq n$. Set $A = \mathrm{Coker}(G_3 \to G_2)$. By applying Lemma 2.7 to the exact sequence $0 \to A \to G_1 \to G_0 \to M \to 0$, we get an exact sequence $0 \to G_n \to \cdots \to G_2 \to G_1' \to P_0 \to M \to 0$ in $\mathrm{Mod}\, S$ with G_1' G_C-projective and P_0 projective. Set $N = \mathrm{Coker}(G_2 \to G_1')$. Then we have $G_C\text{-pd}_S N \leq n - 1$. By the induction hypothesis, there exists an exact sequence $0 \to X_n \to \cdots \to X_t \to \cdots \to X_1 \to P_0 \to M \to 0$ in $\mathrm{Mod}\, S$ such that P_0 is projective and X_t is G_C-projective and $X_i \in \overline{\mathrm{Add}_S C}$ for $i \neq t$ and $1 \leq t \leq n$.

Now we need only to prove (2) for $t = 0$. Set $B = \mathrm{Coker}(G_2 \to G_1)$. By the induction hypothesis, we get an exact sequence $0 \to X_n \to \cdots \to X_3 \to X_2 \to G_1' \to B \to 0$ in $\mathrm{Mod}\, S$ with G_1' G_C-projective and $X_i \in \overline{\mathrm{Add}_S C}$ for any $2 \leq i \leq n$. Set $D = \mathrm{Coker}(X_3 \to X_2)$. Then by applying Lemma 2.7 to the exact sequence $0 \to D \to G_1' \to G_0 \to M \to 0$, we get an exact sequence $0 \to D \to C_1 \to G_0' \to M \to 0$ in $\mathrm{Mod}\, S$ with $C_1 \in \mathrm{Add}_S C$ and G_0' G_C-projective. Thus we obtain the desired exact sequence $0 \to X_n \to \cdots \to X_2 \to X_1 \to G_0' \to M \to 0$ in $\mathrm{Mod}\, S$ with all $X_i \in \overline{\mathrm{Add}_S C}$ and G_0' G_C-projective. □

Let $\mathscr{X}$ a subclass of $\mathrm{Mod}\, S$. Recall from [6] that a homomorphism $f : X \to D$ in $\mathrm{Mod}\, S$ with $X \in \mathscr{X}$ is said to be a *$\mathscr{X}$-precover* of D if for any homomorphism $g : X' \to D$ in $\mathrm{Mod}\, S$ with $X' \in \mathscr{X}$, there exists a homomorphism $h : X' \to X$ such that the following diagram commutes:



Dually, the notion of a *$\mathscr{X}$-preenvelope* is defined. Recall from [12] that a semidualizing bimodule ${}_SC_R$ is called *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R:

(1) If $\mathrm{Hom}_S(C, N) = 0$, then $N = 0$;
(2) If $\mathrm{Hom}_{R^{op}}(C, M) = 0$, then $M = 0$.

Also recall that a ring is called *n-Gorenstein* if it is a left and right Noetherian ring with left and right self-injective dimensions. We denote $\mathscr{I}(S)_{<\infty} = \{L \in \mathrm{Mod}\, S \mid \mathrm{id}_S L < \infty\}$.

Theorem 3.6. *Let ${}_SC_R$ be a faithfully semidualizing bimodule and $n \geq 0$.*

(1) *If S is n-Gorenstein, then $\mathrm{id}_S C \leq n$ if and only if G_C-$\mathrm{pd}_S M \leq n$ for any $M \in \mathrm{Mod}\, S$.*
(2) *If R is n-Gorenstein, then $\mathrm{id}_{R^{op}} C \leq n$ if and only if G_C-$\mathrm{pd}_{R^{op}} N \leq n$ for any $N \in \mathrm{Mod}\, R^{op}$.*

Proof. (1) We first prove the necessity. Let $M \in \mathrm{Mod}\, S$ and

$$0 \to G \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

be an exact sequence in $\mathrm{Mod}\, S$ with all P_i projective. It suffices to show that G is G_C-projective.

Since $\mathrm{id}_S C \leq n$ by assumption, $\mathrm{id}_S C' \leq n$ for any $C' \in \mathrm{Add}_S C$. So $\mathrm{Ext}^i_S(G, \mathrm{Add}_S C) \cong \mathrm{Ext}^{n+i}_S(M, \mathrm{Add}_S C) = 0$ for any $i \geq 1$. By [8, Lemma 10.2.13], G has a monic $\mathscr{I}(S)_{<\infty}$-preenvelope $\alpha : G \rightarrowtail L$. Let $\beta : P^0 \twoheadrightarrow \mathrm{Hom}_S(C, L)$ be a projective precover of $\mathrm{Hom}_S(C, L)$ in $\mathrm{Mod}\, R$. Let γ be the composite homomorphism

$$C \otimes_R P^0 \xrightarrow{1_C \otimes \alpha} C \otimes_R \mathrm{Hom}_S(C, L) \xrightarrow{\nu_L} L$$

with $1_C \otimes \alpha$ epic, where ν_L is the natural evaluation homomorphism. Since $L \in \mathscr{B}_C(S)$ by [12, Corollary 6.2], ν_L is an isomorphism, and γ is epic. Put $C^0 = C \otimes_R P^0$. By an argument similar to that in the proof of [12, Proposition 5.3], we have that $\gamma : C^0 \twoheadrightarrow L$ is a $\mathscr{P}_C(S)$-precover of L. Notice that $\mathrm{Ker}\,\gamma \in \mathscr{I}(S)_{<\infty}$, so $\mathrm{id}_S \mathrm{Ker}\,\gamma \leq n$ by [13, Theorem 2], and hence $\mathrm{Hom}_S(G, C^0) \xrightarrow{\mathrm{Hom}_S(G,\gamma)} \mathrm{Hom}_S(G, L) \to 0$ is exact. It implies that there exists a homomorphism $\delta : G \to C^0$ in $\mathrm{Mod}\, S$ such that $\alpha = \gamma\delta$ and δ is monic.

Let $f : G \to C'$ be a homomorphism in $\mathrm{Mod}\, S$ with $C' \in \mathrm{Add}_S C$. Notice that $C' \in \mathscr{I}(S)_{<\infty}$, so there exists a homomorphism $g : L \to C'$ such that $f = g\alpha$, and hence $f = (g\gamma)\delta$. It implies that $\delta : G \to C^0$ is a monic $\mathrm{Add}_S C$-preenvelope of G. Then $\mathrm{Ext}^i_S(C^0/G, \mathrm{Add}_S C) = 0$ for any $i \geq 1$. Since C^0/G has a monic $\mathscr{I}(S)_{<\infty}$-preenvelope, and so by the above argument it also has a monic $\mathrm{Add}_S C$-preenvelope. We proceed in this manner to get a $\mathrm{Hom}_S(-, \mathrm{Add}_S C)$ exact exact sequence

$$0 \to G \to C^0 \to C^1 \to \cdots$$

in Mod S with all $C^i \in \mathrm{Add}_S C$. It follows from [17, Proposition 2.2] that G is G_C-projective.

Conversely, let $M \in \mathrm{Mod}\, S$ and $0 \to G \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ be an exact sequence in Mod S with all P_i projective. Then G is G_C-projective by assumption. So $\mathrm{Ext}_S^{n+i}(M, C) \cong \mathrm{Ext}_S^i(G, C) = 0$ for any $i \geq 1$. It implies that $\mathrm{id}_S C \leq n$.

(2) It is similar to (1). □

4. THE FOXBY EQUIVALENCE

In this section, ${}_S C_R$ is a faithfully semidualizing bimodule. We will study the Foxby equivalence between the subclasses of the Auslander class $\mathcal{A}_C(R)$ and that of the Bass class $\mathcal{B}_C(S)$.

Lemma 4.1.

(1) *If $M \in \mathcal{A}_C(R)$, then M is G_C-injective if and only if $C \otimes_R M$ is Gorenstein injective.*
(2) *If $M \in \mathcal{B}_C(S)$, then M is G_C-projective if and only if $\mathrm{Hom}_S(C, M)$ is Gorenstein projective.*

Proof. (1) It was proved by Holm and Jørgensen in [11] for commutative rings, see Step 1 in the proof of [11, Theorem 4.2]. The argument there remains valid in our setting.

(2) It is dual to (1). □

Lemma 4.2 ([12, Proposition 4.1]). *There are equivalences of categories*

$$\mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{B}_C(S).$$

The following result shows that the class of Gorenstein projective (resp., G_C-injective) left R-modules in the Auslander class $\mathcal{A}_C(R)$ and the class of G_C-projective (resp., Gorenstein injective) left S-modules in the Bass class $\mathcal{B}_C(S)$ are equivalent under Foxby equivalence.

Proposition 4.3. *There are equivalences of categories*

$$\mathcal{GP}(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{GP}_C(S) \cap \mathcal{B}_C(S),$$

$$\mathcal{GI}_C(R) \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{GI}(S) \cap \mathcal{B}_C(S),$$

Proof. It suffices to prove the first assertion. Dually, we get the second one.

By Lemmas 4.1(2) and 4.2, we have that the functor $\mathrm{Hom}_S(C,-)$ maps $\mathcal{GP}_C(S)\cap\mathcal{B}_C(S)$ to $\mathcal{GP}(R)\cap\mathcal{A}_C(R)$. Next we show that the functor $C\otimes_R -$ maps $\mathcal{GP}(R)\cap\mathcal{A}_C(R)$ to $\mathcal{GP}_C(S)\cap\mathcal{B}_C(S)$.

Let $M\in\mathcal{GP}(R)\cap\mathcal{A}_C(R)$. Then there exists a $\mathrm{Hom}_R(-,\mathrm{Add}_R R)$ exact exact sequence

$$\cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P^0 \xrightarrow{f^1} P^1 \xrightarrow{f^2} \cdots \tag{4.1}$$

in $\mathrm{Mod}\,R$ with all P_i, P^i projective and $M\cong\mathrm{Im}\,f_0$. By [12, Lemma 4.1 and Corollary 6.3], every kernel and cokernel in (4.1) are in $\mathcal{A}_C(R)$. Then, applying the functor $C\otimes_R -$ to (4.1), we get an exact sequence

$$\cdots \xrightarrow{1_C\otimes f_2} C\otimes_R P_1 \xrightarrow{1_C\otimes f_1} C\otimes_R P_0 \xrightarrow{1_C\otimes f_0} C\otimes_R P^0 \xrightarrow{1_C\otimes f^1} C\otimes_R P^1 \xrightarrow{1_C\otimes f^2} \cdots \tag{4.2}$$

in $\mathrm{Mod}\,S$. By [12, Theorem 6.4], we have

$$\mathrm{Ext}^1_S(C\otimes_R \mathrm{Im}\,f_i, C\otimes_R P)\cong\mathrm{Ext}^1_R(\mathrm{Im}\,f_i,P)=0, \qquad \text{and}$$

$$\mathrm{Ext}^1_S(C\otimes_R \mathrm{Im}\,f^i, C\otimes_R P)\cong\mathrm{Ext}^1_R(\mathrm{Im}\,f^i,P)=0$$

for any projective left R-module P and $i\geq 0$. So (4.2) is a $\mathrm{Hom}_S(-,\mathrm{Add}_S C)$ exact exact sequence, and hence $C\otimes_R M\in\mathcal{GP}_C(S)\cap\mathcal{B}_C(S)$ by Lemma 4.2 and Corollary 2.10.

Finally, if $M\in\mathcal{GP}(R)\cap\mathcal{A}_C(R)$ and $N\in\mathcal{GP}_C(S)\cap\mathcal{B}_C(S)$, then there exist natural isomorphisms $M\cong\mathrm{Hom}_S(C,C\otimes_R M)$ and $N\cong C\otimes_R\mathrm{Hom}_S(C,N)$. The proof is finished. □

For any $n\geq 0$, set

$\mathcal{GP}(R)_{\leq n}$ = the class of left R-modules with Gorenstein projective dimension at most n,

$\mathcal{GI}(S)_{\leq n}$ = the class of left S-modules with Gorenstein injective dimension at most n,

$\mathcal{GP}_C(S)_{\leq n}$ = the class of left S-modules with G_C-projective dimension at most n,

$\mathcal{GI}_C(R)_{\leq n}$ = the class of left R-modules with G_C-injective dimension at most n.

As a consequence of Proposition 4.3, we get the following result. The commutative version of this result was proved in [15, Remark 2.11].

Theorem 4.4. *For any $n\geq 0$, there are equivalences of categories*

$$\mathcal{GP}(R)_{\leq n}\cap\mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{GP}_C(S)_{\leq n}\cap\mathcal{B}_C(S).$$

$$\mathcal{GI}_C(R)_{\leq n} \cap \mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{GI}(S)_{\leq n} \cap \mathcal{B}_C(S).$$

Proof. It suffices to prove the first assertion. Dually, we get the second one.

Let $M \in \mathcal{GP}(R)_{\leq n} \cap \mathcal{A}_C(R)$. If $n = 0$, then $C \otimes_R M \in \mathcal{GP}_C(S) \cap \mathcal{B}_C(S)$ by Proposition 4.3. Now suppose $n \geq 1$. Then by Proposition 3.5 with $C = R$, there exists an exact sequence

$$0 \to P_n \to \cdots \to P_2 \to P_1 \to G_0 \to M \to 0 \tag{4.3}$$

in Mod R with G_0 Gorenstein projective and P_i projective for any $1 \leq i \leq n$. By [12, Corollaries 6.2 and 6.3], every term and every cokernel in (4.3) are in $\mathcal{A}_C(R)$. So we get an exact sequence

$$0 \to C \otimes_R P_n \to \cdots C \otimes_R P_2 \to C \otimes_R P_1 \to C \otimes_R G_0 \to C \otimes_R M \to 0$$

in Mod S with $C \otimes_R G_0$ and all $C \otimes_R P_i$ in $\mathcal{GP}_C(S) \cap \mathcal{B}_C(S)$ by Proposition 4.3. Thus $C \otimes_R M \in \mathcal{GP}_C(S)_{\leq n} \cap \mathcal{B}_C(S)$.

Conversely, let $M \in \mathcal{GP}_C(S)_{\leq n} \cap \mathcal{B}_C(S)$. If $n = 0$, then $\mathrm{Hom}_S(C, M) \in \mathcal{GP}(R) \cap \mathcal{A}_C(R)$ by Proposition 4.3. Now suppose $n \geq 1$. Then by [17, Theorem 3.6], there exists an exact sequence

$$0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to G_0 \to M \to 0 \tag{4.4}$$

in Mod S with $C_i \in \mathrm{Add}_S C$ for any $1 \leq i \leq n$ and G_0 G_C-projective. By [12, Corollaries 6.1 and 6.3], every term and every cokernel in (4.4) are in $\mathcal{B}_C(S)$. So, applying the functor $\mathrm{Hom}_S(C, -)$ to (4.4), we get an exact sequence

$$0 \to \mathrm{Hom}_S(C, C_n) \to \cdots \to \mathrm{Hom}_S(C, C_1) \to \mathrm{Hom}_S(C, G_0) \to \mathrm{Hom}_S(C, M) \to 0$$

in Mod R with $\mathrm{Hom}_S(C, G_0)$ and all $\mathrm{Hom}_S(C, C_i)$ in $\mathcal{GP}(R) \cap \mathcal{A}_C(R)$ by Proposition 4.3. Thus $M \in \mathcal{GP}(R)_{\leq n} \cap \mathcal{A}_C(R)$. □

For any $n \geq 0$, we denote by $\mathcal{P}(R)_{\leq n}$ (resp., $\mathcal{I}(S)_{\leq n}$) the class of left R-modules (resp., left S-modules) with projective dimension (resp., injective dimension) at most n and by $\mathcal{P}_C(S)_{\leq n}$ (resp., $\mathcal{I}_C(R)_{\leq n}$) the class of left S-modules (resp. left R-modules) with C-projective dimension (resp., C-injective dimension) at most n. The commutative version of the following result was proved in [16, Theorem 2.12].

Proposition 4.5. *For any $n \geq 0$, there are equivalences of categories*

$$\mathcal{P}(R)_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{P}_C(S)_{\leq n},$$

$$\mathcal{I}_C(R)_{\leq n} \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{I}(S)_{\leq n}.$$

Proof. Let $n \geq 0$. We have that $\mathscr{P}(R)_{\leq n} \subseteq \mathscr{A}_C(R)$ and $\mathscr{I}(S)_{\leq n} \subseteq \mathscr{B}_C(S)$ by [12, Corollary 6.2]. On the other hand, $\mathscr{P}_C(S)_{\leq n} \subseteq \mathscr{B}_C(S)$ and $\mathscr{I}_C(R)_{\leq n} \subseteq \mathscr{A}_C(R)$ by [12, Corollary 6.1]. Then the assertions follow easily. □

Putting the results in this section together, we get the following theorem.

Theorem 4.6 (Foxby Equivalence). *For any $n \geq 0$, there are equivalences of categories*:

$$
\begin{array}{ccc}
\mathscr{P}(R)_{\leq n} & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathscr{P}_C(S)_{\leq n} \\
\downarrow & & \downarrow \\
\mathscr{G}\mathscr{P}(R)_{\leq n} \cap \mathscr{A}_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathscr{G}\mathscr{P}_C(S)_{\leq n} \cap \mathscr{B}_C(S) \\
\downarrow & & \downarrow \\
\mathscr{A}_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathscr{B}_C(S) \\
\uparrow & & \uparrow \\
\mathscr{G}\mathscr{I}_C(R)_{\leq n} \cap \mathscr{A}_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathscr{G}\mathscr{I}(S)_{\leq n} \cap \mathscr{B}_C(S) \\
\uparrow & & \uparrow \\
\mathscr{I}_C(R)_{\leq n} & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \sim & \mathscr{I}(S)_{\leq n}.
\end{array}
$$

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